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The Scattering Matrix of a General Interconnection of Multipoles

by

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THE SCATTERING MATRIX OF A GENERAL INTERCONNECTION OF MULTIPOLES

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1. Introduction

In the most general case, a multipole is an electrical network with several pairs of terminals. If the dimensions of a network become large enough to be comparable to the wavelength, and if the circuits used are general waveguides rather than twin (loop) circuits, a description in terms of currents and voltages at the terminal pairs is no longer meaningful. In this case especially, the use of so-called "wave matrices" is recommended, and for networks with more than two pairs of terminals in particular, the use of the scattering matrix. It relates the normalized waves running outward from the terminal pairs of a network to the waves running toward the terminals from the outside and does so in the planes of the terminal pairs. For the purpose of the following article, a basic familiarity with this descriptive method is assumed (Ref. 1). Only assumptions and notations will be briefly explained.

Under the following conditions, the description of a network in terms of the scattering matrix is possible and meaningful:

Condition a: The network must be linear.

Condition b: The network must not contain any generators.

In terminology appropriate for a scattering matrix: the network must not have an outgoing wave at any of the terminal pairs unless it has an incoming wave at least at one terminal pair.

Condition c: Only the reflectivity and propagation properties of the multipole are described for *very specific* "predetermined" types of waves in a *specific* "predetermined" circuit at the individual pairs of terminals. These "predetermined" circuits and wave types may differ from terminal pair to terminal pair.

From this, it follows that the scattering matrix determined for a multipole can only reproduce the physical behavior of the multipole accurately if it is ensured by some means that only the "predetermined" wave types actually appear at the terminal pairs. It therefore is senseless to cite an extreme example to treat the interconnection of the coaxial-circuit terminal pairs of a multipole with the waveguide terminal pair of another multipole with the aid of scattering matrices which were presupposed to have a coaxial TEM-wave for the coaxial terminal pair and a TE_{10}-wave for the waveguide terminal pair. It is not even possible for a coaxial TEM wave to appear exclusively at the site of the connected pair of terminals with a TE_{10}-wave exclusively appearing at the other pair. Thus, for the connection of two terminal pairs, the following holds:

Condition d: The connection of two pairs of terminals is accurately described only by scattering matrices that are identical with respect to the "predetermined" circuit and the "predetermined" wave types in this circuit.

Condition e: In addition to this necessary condition, one must make sure that, after the interconnection, only these "predetermined" wave types appear.

Although it is possible to predetermine several wave types per pair of terminals (in the scattering matrix, this terminal pair then shows as an apparent multiple, that is, once per wave type), though only one circuit, it will be assumed for what follows that only *one* wave type is admitted per terminal pair. This case occurs most frequently in practice. In this case, condition e) is satisfied by the existence of sufficiently long line elements, which lead into the interior of the multipole and on which only the "predetermined" wavetype can propagate as the so-called "principal wave."

Figure 1 shows the νth terminal pair of a multipole with the outgoing and incoming waves:

normalized incoming wave at terminal pair ν: $\underline{a}_\nu$,

normalized outgoing wave at terminal pair ν: $\underline{b}_\nu$,

The normalizations are selected so that

$$\frac{1}{2}|\underline{a}_\nu|^2 = P_\nu^{zu}$$

is the power of the incoming wave and

$$\frac{1}{2}|\underline{b}_\nu|^2 = P_\nu^{ab}$$

the power of the outgoing wave. Complex magnitudes are underlined, matrices are boldface type.

The problem of determining the scattering matrix of a general interconnection given there, however, is not entirely satisfactory in two respects:

1. It does not supply a closed term for the scattering matrix of the new multipole.

2. Its accuracy has been proven only by examples thus far.

The interconnection of two partial multipoles with an arbitrary number of terminal pairs is treated in Ref. 2. The formula is more closed but very complicated. Special cases of multipole interconnections are computed in Ref. 3. The purpose of this study is to derive a *closed*, *comprehensive* term for the scattering matrix of the new multipole and to prove the accuracy of the formula presented in Ref. 1.

2. The Interconnection of Multipoles

The most common interconnection of given individual[1] multipoles with a new multipole consist of

I. shutting off several terminal pairs of the individual multipoles with specific reflection coefficients,

II. interconnecting individual terminal pairs of the individual multipoles, perhaps across trunk circuits.

III. constructing the terminal pairs of the new multipole from the remaining terminal pairs - perhaps using extension circuits.

Let the individual multipoles be given by their scattering matrices. All conditions cited in the introduction for the multipole (conditions a and b), for the scattering matrices (conditions c)[2], and for the interconnection (conditions d and e) shall be satisfied.

[1] Naturally, this can only be a single multipole that has been wired.

[2] With the above-mentioned constraint of *one* wave type per terminal pair.

Next, the total of the unwired individual multipoles is combined into a large 2n-pole. The terminal pairs will be designated as follows:

The terminal pairs given under III. are assigned numbers 1 to m in arbitrary sequence. Later, extended across circuits, they form the terminal pairs of the new multipole, which is therefore a 2m-pole. The terminal pairs given under II. terminal pair $m+1, m+2, \ldots,$ is such a sequence that terminal pair $m+1$ will later be connected with $m+2$, terminal pair $m+3$ with $m+4$, etc.

The terminal pairs given under I. receive the remaining numbers up to n, in arbitrary sequence.

Furthermore, all terminal pairs are given specific counting arrows for the incoming and outgoing waves $\underline{a}_\nu$ and $\underline{b}_\nu$. Figure 2 shows such a 2n-pole schematically.

The associated matrix is

$$\begin{pmatrix} \underline{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{b}_m \\ \underline{b}_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{b}_n \end{pmatrix} = \underline{S} \begin{pmatrix} \underline{a}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_m \\ \underline{a}_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_n \end{pmatrix}. \qquad (1)$$

Its elements are determined by the given scattering matrices of the individual multipoles.

The total matrix $\underline{S}$ is divided into four submatrices (Ref. 6) in the following manner:

$$\left(\begin{array}{c} \underline{b}_1 \\ \cdot \\ \cdot \\ \underline{b}_m \\ \hline \underline{b}_{m+1} \\ \cdot \\ \cdot \\ \underline{b}_n \end{array}\right) = \left(\begin{array}{c|c} \underline{S}_1 & \underline{S}_2 \\ \hline \underline{S}_3 & \underline{S}_4 \end{array}\right) \cdot \left(\begin{array}{c} \underline{a}_1 \\ \cdot \\ \cdot \\ \underline{a}_m \\ \hline \underline{a}_{m+1} \\ \cdot \\ \cdot \\ \underline{a}_n \end{array}\right). \qquad (2)$$

with the abbreviations

$$\begin{pmatrix} \underline{b}_1 \\ \cdot \\ \cdot \\ \underline{b}_m \end{pmatrix} = \underline{b}_m, \quad \begin{pmatrix} \underline{a}_1 \\ \cdot \\ \cdot \\ \underline{a}_m \end{pmatrix} = \underline{a}_m,$$
$$\begin{pmatrix} \underline{b}_{m+1} \\ \cdot \\ \cdot \\ \underline{b}_n \end{pmatrix} = \underline{b}_n, \quad \begin{pmatrix} \underline{a}_{m+1} \\ \cdot \\ \cdot \\ \underline{a}_n \end{pmatrix} = \underline{a}_n \qquad (3)$$

then,

$$b_m = S_1 a_m + S_2 a_n,$$
$$b_n = S_3 a_m + S_4 a_n. \qquad (4)$$

The $2n$-pole is completely described by these equations.

In a similar manner, the total of the shut-offs mentioned under I. and the circuit quadri-poles given in II. are combined into a second large $2(n + m)$-pole. Later, n terminal pairs from this $2(n + m)$-pole are connected with the $2n$-pole, and the remaining terminal pairs become the terminal pairs of the new $2m$-pole. The terminal pairs receive the following designation here:

The n terminal pairs that will later be connected with the $2n$-pole are designated 1' to n', in such a sequence that terminal pair v' is later connected with terminal pair v.

The remaining m terminal pairs shall be designated 1" to m'', in such a sequence that terminal pair v'' is connected with v' within the $2(n + m)$-pole by an extension circuit. Terminal pairs 1" to m'' will later form the terminal pairs of the new $2m$-pole.

Furthermore, the counting arrows for the incoming and outgoing waves $\underline{a}_{v'}$ and $\underline{b}_{v'}$ for all terminal pairs v' are combined in such a way that, when the interconnection is made later, they are parallel to the counting arrows on terminal pairs v, as shown in Figure 3.

The counting arrows on terminal pairs v'' for the incoming and outgoing waves $\underline{a}_{v''}$ and $\underline{b}_{v''}$ should point in the direction of the counting arrows at the other end of the extension circuit according to Fig. 4. Figure 5 schematically shows such a $2(n + m)$-pole with internal connecting circuits, extension circuits, and shut-offs. To make the arrangement of the counting arrows and the designation of the terminal pairs more apparent, the 2n-pole is also shown.

According to Fig. 4, for a couple of terminal pairs ν' and $\nu''(\nu = 1, 2, \ldots, m)$ connected by an extension circuit,

$$\begin{pmatrix} \underline{b}_{\nu'} \\ \underline{b}_{\nu''} \end{pmatrix} = \begin{pmatrix} 0 & e^{-\underline{g}_\nu} \\ e^{-\underline{g}_\nu} & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{a}_{\nu'} \\ \underline{a}_{\nu''} \end{pmatrix},$$

where $\underline{g}_\nu$ is the propagation constant of the νth extension circuit. For the total of terminal pairs 1' to m' and 1" to m", then,

$$\begin{bmatrix} \underline{b}_{1'} \\ \vdots \\ \underline{b}_{m'} \\ \hline \underline{b}_{1''} \\ \vdots \\ \underline{b}_{m''} \end{bmatrix} = \left[\begin{array}{c|c} (0) & \underline{K}_1 \\ \hline \underline{K}_1 & (0) \end{array}\right] \cdot \begin{bmatrix} \underline{a}_{1'} \\ \vdots \\ \underline{a}_{m'} \\ \hline \underline{a}_{1''} \\ \vdots \\ \underline{a}_{m''} \end{bmatrix}, \quad (5)$$

with

$$\underline{K}_1 = \begin{bmatrix} e^{-\underline{g}_1} & 0 & \ldots & 0 \\ 0 & e^{-\underline{g}_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 0 & \ldots & 0 & e^{-\underline{g}_m} \end{bmatrix}. \quad (6)$$

The abbreviations

$$\begin{bmatrix} \underline{b}_{1'} \\ \vdots \\ \underline{b}_{m'} \end{bmatrix} = \underline{b}_{m'}, \quad \begin{bmatrix} \underline{a}_{1'} \\ \vdots \\ \underline{a}_{m'} \end{bmatrix} = \underline{a}_{m'},$$
$$\begin{bmatrix} \underline{b}_{1''} \\ \vdots \\ \underline{b}_{m''} \end{bmatrix} = \underline{b}_{m''}, \quad \begin{bmatrix} \underline{a}_{1''} \\ \vdots \\ \underline{a}_{m''} \end{bmatrix} = \underline{a}_{m''} \quad (7)$$

yield the equations

$$\underline{b}_{m'} = \underline{K}_1 \underline{a}_{m''}, \quad \underline{b}_{m''} = \underline{K}_1 \underline{a}_{m'}. \quad (8)$$

For a terminal-pair couple ν' and $(\nu + 1)'$ $(\nu = m + 1, m + 3, \ldots)$ connected with the $2(n + m)$-pole by a connecting circuit, according to Fig. 6, either

$$\begin{pmatrix} a_{\nu'} \\ a_{(\nu+1)'} \end{pmatrix} = \begin{pmatrix} 0 & e^{g_\nu} \\ e^{g_\nu} & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{\nu'} \\ b_{(\nu+1)'} \end{pmatrix} \qquad \textit{(case 1)}$$

or

$$\begin{pmatrix} a_{\nu'} \\ a_{(\nu+1)'} \end{pmatrix} = \begin{pmatrix} 0 & -e^{g_\nu} \\ -e^{g_\nu} & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{\nu'} \\ b_{(\nu+1)'} \end{pmatrix}. \qquad \textit{(case 2)}$$

where g_ν is the propagation constant of the νth connecting circuit.

According to Fig. 7, for a terminal pair $\nu'(\nu = n, n - 1, \ldots)$ wired in the $2(n + m)$-pole with a specific reflection coefficient,

$$a_{\nu'} = \frac{1}{r_\nu} b_{\nu'}.$$

where r_ν is the reflection coefficient of the shut-off. In the case of $r_\nu = 0$, the above relation is not valid. This difficulty is solved in the following manner:

If a terminal pair of the 2n-pole is shut off reflection-free, then (see Fig. 2)

$$a_\nu = 0.$$

Since the outgoing wave b_ν is not of interest on this terminal pair, the reflection-free shut-off can be expressed by deleting the νth row and the νth column in scattering matrix $\underline{S}$ (Eq. 1). Terminal pair ν then will not appear in subsequent calculations. In this way, case $r_\nu = 0$ for Fig. 7 is excluded, and the need to consider limitations is circumvented.

For the total of terminal pairs $(m + 1)'$ to n', now

$$\begin{bmatrix} a_{(m+1)'} \\ \cdot \\ \cdot \\ \cdot \\ a_{n'} \end{bmatrix} = \underline{K}_2 \begin{bmatrix} b_{(m+1)'} \\ \cdot \\ \cdot \\ \cdot \\ b_{n'} \end{bmatrix} \qquad (9)$$

with

$$
\underline{K}_2 = \begin{pmatrix}
0 & \pm e^{g_{m+1}} & 0 & 0 & \cdots & 0 \\
\pm e^{g_{m+1}} & 0 & 0 & 0 & & \cdot \\
0 & 0 & 0 & \pm e^{g_{m+3}} & & \cdot \\
0 & 0 & \pm e^{g_{m+3}} & 0 & \ddots & \cdot \\
\cdot & & & \ddots & & \cdot \\
\cdot & & & & \frac{1}{r_{n-1}} & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{1}{r_n}
\end{pmatrix} \qquad (10).
$$

The abbreviations

$$
\begin{bmatrix} \underline{a}_{(m+1)'} \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_{n'} \end{bmatrix} = \underline{a}_{n'}, \qquad \begin{bmatrix} \underline{b}_{(m+1)'} \\ \cdot \\ \cdot \\ \cdot \\ \underline{b}_{n'} \end{bmatrix} = \underline{b}_{n'} \qquad (11)
$$

yield

$$
\underline{a}_{n'} = \underline{K}_2 \underline{b}_{n'}. \qquad (12)
$$

The 2(n + m)-pole is completely described by Eqs. (8) and (12).

The interconnection of terminal pairs ν with terminal pairs ν' (ν = 1,2,...n) yields, according to Fig. 3,

$$
\begin{aligned} \underline{a}_{\nu'} &= \underline{b}_{\nu}, \\ \underline{b}_{\nu'} &= \underline{a}_{\nu}, \end{aligned} \qquad \nu = 1, 2, \ldots, n.
$$

From this, it follows that

$$
\begin{aligned} \underline{a}_{m'} &= \underline{b}_{m}, & \underline{a}_{n'} &= \underline{b}_{n}, \\ \underline{b}_{m'} &= \underline{a}_{m}, & \underline{b}_{n'} &= \underline{a}_{n}. \end{aligned} \qquad (13)
$$

These equations describe the fact of the interconnection.

Equations (4), (8), (12), and (13) form a system of linear matrix equations, which must be reduced according to

$$
\underline{b}_{m''} = \underline{S}'' \underline{a}_{m''} \qquad (14)
$$

$\underline{S}''$ is then the scattering matrix of the new 2m-pole. The reduction of the system is carried out in the appendix. The result is

$$
\underline{S}'' = \underline{K}_1 [\underline{S}_1 - \underline{S}_2 (\underline{S}_4 - \underline{K}_2)^{-1} \underline{S}_3] \underline{K}_1. \qquad (15)
$$

This relation represents a generally valid, closed, and comprehensive expression for scattering matrix $\underline{S}''$ of the new multipole.

3. Discussion of Eq. (15).

The effect of the extension circuit is extressed by K_1. If a 2n-pole is wired exclusively with extension circuits, the scattering matrix of the new multipole, from Eq. (15), is:

$$\underline{S}'' = \underline{K}_1 \underline{S} \underline{K}_1. \qquad (16)$$

In this case,

$$\underline{S}_1 = \underline{S}.$$

Accordingly (see Eq. 6 for $\underline{K}_1$), element $\underline{S}_{\mu''\nu''}$ of the new scattering matrix is

$$\underline{S}_{\mu''\nu''} = \underline{S}_{\mu\nu} e^{-(g_\mu + g_\nu)}. \qquad (17)$$

Equations (16) and (17) are found in Refs. 1 and 4.

Since the effect of extension circuits is easy to overlook, as in Eq. (17), these will generally not to be taken into consideration. Matrix $\underline{K}_1$ then becomes a unit matrix, and Eq. (15) adopts the simpler form

$$\underline{S}'' = \underline{S}_1 - \underline{S}_2(\underline{S}_4 - \underline{K}_2)^{-1} \underline{S}_3. \qquad (18)$$

From this, we obtain for element $\underline{S}_{\mu''\nu''}$

$$\underline{S}_{\mu''\nu''} = \underline{S}_{1\mu\nu} - {}^{\mu}\underline{S}_2(\underline{S}_4 - \underline{K}_2)^{-1} {}_{\nu}\underline{S}_3. \qquad (19)$$

where

$\underline{S}_{1\mu\nu}$ is the element in the μth row and the νth column of matrix $\underline{S}_1$,

${}^{\mu}\underline{S}_2$ is the row matrix consisting of the μth row of $\underline{S}_2$,

${}_{\nu}\underline{S}_3$ is the column matrix consisting of the νth column of $\underline{S}_3$.

As shown in the appendix, Eq. (19) can be stated in the form

$$\underline{S}_{\mu''\nu''} = \frac{\det\left(\begin{array}{c|c} \underline{S}_{1\mu\nu} & {}^{\mu}\underline{S}_2 \\ \hline {}_{\nu}\underline{S}_3 & \underline{S}_4 - \underline{K}_4 \end{array}\right)}{\det(\underline{S}_4 - \underline{K}_4)} \qquad (20)$$

Equation (20) represents - in a somewhat changed notation - the relations given in Ref. 1 for a single element $\underline{S}_{\mu''\nu''}$ of the new scattering matrix $\underline{S}''$. The general validity of the formula presented there is thus demonstrated.

A comparison of Eq. (20) with Eq. (18) with respect to the amount of the calculation needed to determine scattering matrix $\underline{S}''$ indicates that

Eq. (20) is preferable if $m < n - m$,

Eq. (18) is preferable if $m > n - m$.

If matrix $\underline{S}_4$ becomes a null matrix, Eq. (18) is decidedly preferred, since then

$$\underline{S}'' = \underline{S}_1 + \underline{S}_2 \underline{K}_2^{-1} \underline{S}_3 \qquad (21)$$

A glance at Eq. (10) shows that the reciprocal matrix

$$\underline{K}_2^{-1} = \begin{pmatrix} 0 & \pm e^{-\underline{g}_{m+1}} & 0 & 0 & \dots & 0 \\ \pm e^{-\underline{g}_{m+1}} & 0 & 0 & 0 & & \cdot \\ 0 & 0 & 0 & \pm e^{-\underline{g}_{m+3}} & & \cdot \\ 0 & 0 & \pm e^{-\underline{g}_{m+3}} & 0 & \ddots & \cdot \\ \cdot & & & \ddots & & \cdot \\ \cdot & & & & \ddots & \cdot \\ \cdot & & & & \underline{r}_{n+1} & 0 \\ 0 & \dots & \dots & \dots & 0 & \underline{r}_n \end{pmatrix} \qquad (22)$$

is easy to derive. Example 8 in Ref. 1 presents such a special case.

If all the terminal pairs of a multipole, except for one terminal pair ($v = 1$), are wired with specific reflection coefficients, Matrix $\underline{K}_2$ has the following form according to Eq. (10):

$$\underline{K}_2 = \begin{pmatrix} \frac{1}{\underline{r}_2} & 0 & \dots & 0 \\ 0 & \frac{1}{\underline{r}_3} & & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & & \cdot \\ 0 & \dots & \dots & \frac{1}{\underline{r}_n} \end{pmatrix}.$$

This matrix can also be written as

$$\underline{K}_2 = \begin{pmatrix} \underline{r}_2 & 0 & \dots & 0 \\ 0 & \underline{r}_3 & & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & & \cdot \\ 0 & \dots & \dots & \underline{r}_n \end{pmatrix}^{-1} = \underline{r}^{-1}.$$

The incoming reflection coefficient of the bipole resulting from the wiring is obtained from this and, with Eq. (19), becomes

$$\underline{r}_{1''E} = \underline{S}_{1''1''} = \underline{S}_{11} - {}^{1}\underline{S}_2(\underline{S}_4 - \underline{r}^{-1})^{-1}\,{}_{1}\underline{S}_3. \quad (23)$$

For the case of the quadripole, this relation takes the well known form

$$\underline{r}_{1''E} = \underline{S}_{11} - \underline{S}_{12}(\underline{S}_{22} - \underline{r}_2^{-1})^{-1}\underline{S}_{21}$$

Equation (23) represents a correspondingly generalized expression for the multipoles.

4. Example.

The scattering matrix for the microwave bridge shown in Fig. 8 is to be determined.

The bridge consists of two ideal magic T's with terminal pairs 1; 3; 5; 7; and 2; 4; 9; 11 and two general quadripoles with terminal pairs 6; 10 and 8; 12. Terminal pairs 5 and 6, 7 and 8, 9 and 10, 11 and 12 are connected. The partial multipoles have scattering matrices (Ref. 1)

$$\begin{array}{c|cccc} & 1 & 3 & 5 & 7 \\ \hline 1 & 0 & 0 & c & c \\ 3 & 0 & 0 & -c & c \\ 5 & c & -c & 0 & 0 \\ 7 & c & c & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 2 & 4 & 11 & 9 \\ \hline 2 & 0 & 0 & c & c \\ 4 & 0 & 0 & -c & c \\ 11 & c & -c & 0 & 0 \\ 9 & c & c & 0 & 0 \end{array} \quad \text{with } c = 1/\sqrt{2}, \quad \begin{array}{c|cc} & 6 & 10 \\ \hline 6 & \underline{S}_{66} & \underline{S}_{610} \\ 10 & \underline{S}_{106} & \underline{S}_{1010} \end{array} \qquad \begin{array}{c|cc} & 8 & 12 \\ \hline 8 & \underline{S}_{88} & \underline{S}_{812} \\ 12 & \underline{S}_{128} & \underline{S}_{1212} \end{array}$$

From this, matrix $\underline{S}$ of Eq. (1) is obtained as

$$\begin{array}{c|cccc|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 1 & 0 & 0 & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & c & 0 \\ 3 & 0 & 0 & 0 & 0 & -c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & -c & 0 \\ \hline 5 & c & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & \underline{S}_{66} & 0 & 0 & 0 & \underline{S}_{610} & 0 & 0 \\ 7 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{S}_{88} & 0 & 0 & 0 & \underline{S}_{812} \\ 9 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & \underline{S}_{106} & 0 & 0 & 0 & \underline{S}_{1010} & 0 & 0 \\ 11 & 0 & c & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{S}_{128} & 0 & 0 & 0 & \underline{S}_{1212} \end{array} = \underline{S}.$$

Matrix $\underline{K}_1$ is the unit matrix, and matrix $\underline{K}_2$, with direct connections and the counting arrows as drawn, has the form (Eq. 10)

$$\begin{array}{c|cccccccc} & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} = \underline{K}_2.$$

The formation of the reciprocal matrix $(\underline{S}_4 - \underline{K}_2)^{-1}$ is very simple and yields

$$
\begin{array}{c|cccccccc}
 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
5 & \underline{S}_{66} & 1 & 0 & 0 & \underline{S}_{610} & 0 & 0 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & \underline{S}_{88} & 1 & 0 & 0 & \underline{S}_{812} & 0 \\
8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
9 & \underline{S}_{106} & 0 & 0 & 0 & \underline{S}_{1010} & 1 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
11 & 0 & 0 & \underline{S}_{128} & 0 & 0 & 0 & \underline{S}_{1212} & 1 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
= -(\underline{S}_4 - \underline{K}_2)^{-1}.
$$

Through elementary matrix multiplication, the new scattering matrix corresponding to Eq. (18) is obtained:

$$
\underline{S}'' = \frac{1}{2}\begin{pmatrix}
(\underline{S}_{66} + \underline{S}_{88}) & (\underline{S}_{610} + \underline{S}_{812}) & (-\underline{S}_{66} + \underline{S}_{88}) & (\underline{S}_{610} - \underline{S}_{812}) \\
(\underline{S}_{106} + \underline{S}_{128}) & (\underline{S}_{1010} + \underline{S}_{1212}) & (-\underline{S}_{106} + \underline{S}_{128}) & (\underline{S}_{1010} - \underline{S}_{1212}) \\
(-\underline{S}_{66} + \underline{S}_{88}) & (-\underline{S}_{610} + \underline{S}_{812}) & (\underline{S}_{66} + \underline{S}_{88}) & (-\underline{S}_{610} - \underline{S}_{812}) \\
(\underline{S}_{106} - \underline{S}_{128}) & (\underline{S}_{1010} - \underline{S}_{1212}) & (-\underline{S}_{106} - \underline{S}_{128}) & (\underline{S}_{1010} + \underline{S}_{1212})
\end{pmatrix}
$$

For the special quadripoles

$$
\begin{array}{c|cc}
 & 6 & 10 \\
\hline
6 & 0 & -1 \\
10 & 1 & 0
\end{array}
\qquad\qquad
\begin{array}{c|cc}
 & 8 & 12 \\
\hline
8 & 0 & 1 \\
12 & 1 & 0
\end{array}
$$

(gyrator) (direct connection)

we obtain the circulator

$$
\underline{S}'' = \begin{bmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}.
$$

If, on the other hand, terminal pairs 2 and 3 are shut off reflection-free, then for the remaining quadripole,

$$
\begin{pmatrix} \underline{b}_1 \\ \underline{b}_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\underline{S}_{66} + \underline{S}_{88}) & (\underline{S}_{610} - \underline{S}_{812}) \\ (\underline{S}_{106} - \underline{S}_{128}) & (\underline{S}_{1010} + \underline{S}_{1212}) \end{pmatrix} \cdot \begin{pmatrix} \underline{a}_1 \\ \underline{a}_4 \end{pmatrix}
$$

This is a microwave measuring bridge. It makes the measurement of nonreciprocal quadripoles (8; 12) through comparison with a quadripole constructed of a variable attenuator and a variable phase shifter (6; 10). The balance condition is obtained, for example, when terminal pair 1 is being charged and pair 4 is shut off with a reflection-free detector over

$$\underline{b}_4 = \frac{1}{2}(\underline{S}_{106} - \underline{S}_{128})\underline{a}_1 = 0$$

to

$$S_{128} = S_{106}$$

It is worthy of note that the measurement result is not falsified by mismatching of the quadripole (8; 12).

Appendix

A.1. Reduction of the linear system of matrix equations.

The following applied:

$$\underline{b}_m = \underline{S}_1 \underline{a}_m + \underline{S}_2 \underline{a}_n, \quad \underline{b}_n = \underline{S}_3 \underline{a}_m + \underline{S}_4 \underline{a}_n; \quad (4)$$

$$\underline{b}_{m'} = \underline{K}_1 \underline{a}_{m''}, \quad \underline{b}_{m''} = \underline{K}_1 \underline{a}_{m'}; \quad (8)$$

$$\underline{a}_{n'} = \underline{K}_2 \underline{b}_{n'}; \quad (12)$$

$$\underline{a}_{m'} = \underline{b}_m, \quad \underline{b}_{m'} = \underline{a}_m, \quad \underline{a}_{n'} = \underline{b}_n, \quad \underline{b}_{n'} = \underline{a}_n. \quad (13)$$

Equations (13), introduced into (8) and (12), yield

$$\underline{a}_m = \underline{K}_1 \underline{a}_{m''}, \quad \underline{b}_{m''} = \underline{K}_1 \underline{b}_m; \quad (8a)$$

$$\underline{b}_n = \underline{K}_2 \underline{a}_n. \quad (12a)$$

Since K_1 never becomes a null matrix, the first expression of Eq. (4) can be expanded with K_1:

$$\underline{K}_1 \underline{b}_m = \underline{K}_1 \underline{S}_1 \underline{a}_m + \underline{K}_1 \underline{S}_2 \underline{a}_n.$$

Introduced into Eq. (4), Eqs. (8a) and (12a) then result in

$$\begin{aligned} \underline{b}_{m''} &= \underline{K}_1 \underline{S}_1 \underline{K}_1 \underline{a}_{m''} + \underline{K}_1 \underline{S}_2 \underline{a}_n, \\ \underline{K}_2 \underline{a}_n &= \underline{S}_3 \underline{K}_1 \underline{a}_{m''} + \underline{S}_4 \underline{a}_n. \end{aligned} \quad (4a)$$

The second expression of Eq. (4a) can be solved for a_n if matrix $S_4 - K_2$ is not singular; i.e., if

$$\det(\underline{S}_4 - \underline{K}_2) \neq 0.$$

The result is

$$\underline{a}_n = -(\underline{S}_4 - \underline{K}_2)^{-1} \underline{S}_3 \underline{K}_1 \underline{a}_{m''}.$$

From this, it follows that

$$\underline{b}_{m''} = \underline{K}_1 [\underline{S}_1 - \underline{S}_2 (\underline{S}_4 - \underline{K}_2)^{-1} \underline{S}_3] \underline{K}_1 \underline{a}_{m''},$$

which proves Eq. (15).

A.2. Proof for the relation (Eq 20) presented in Ref. 1.

The identity

$$\frac{\det\left(\begin{array}{c|c} \underline{S}_{1\mu\nu} & {}^{\mu}\underline{S}_2 \\ \hline {}_{\nu}\underline{S}_3 & \underline{S}_4-\underline{K}_2 \end{array}\right)}{\det(\underline{S}_4-\underline{K}_2)} = \underline{S}_{1\mu\nu} - {}^{\mu}\underline{S}_2(\underline{S}_4-\underline{K}_2)^{-1}\,{}_{\nu}\underline{S}_3$$

must be proved. The reciprocal matrix

$$\left(\begin{array}{c|c} \underline{S}_{1\mu\nu} & {}^{\mu}\underline{S}_2 \\ \hline {}_{\nu}\underline{S}_3 & \underline{S}_4-\underline{K}_2 \end{array}\right)\left(\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right) = \underline{E}.$$

is constructed. The result is

$$\underline{S}_{1\mu\nu}\,\underline{A} + {}^{\mu}\underline{S}_2\,\underline{C} = 1,$$

$${}_{\nu}\underline{S}_3\,\underline{A} + (\underline{S}_4-\underline{K}_2)\,\underline{C} = (0).$$

It follows from this that

$$(\underline{S}_{1\mu\nu} - {}^{\mu}\underline{S}_2(\underline{S}_4-\underline{K}_2)^{-1}\,{}_{\nu}\underline{S}_3)\,\underline{A} = 1.$$

On the other hand, as we know, for the element A we obtain

$$\underline{A} = \frac{\det(\underline{S}_4-\underline{K}_2)}{\det\left(\begin{array}{c|c} \underline{S}_{1\mu\nu} & {}^{\mu}\underline{S}_2 \\ \hline {}_{\nu}\underline{S}_3 & \underline{S}_4-\underline{K}_2 \end{array}\right)}.$$

This constitutesthe proof. The identity presented is a special case of Sylvester's law (Ref. 5).

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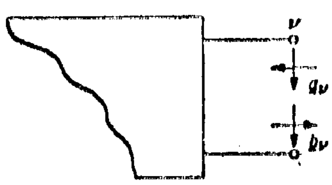

Bild 1.
Zu- und ablaufende Wellen am ν-ten Klemmenpaar.

Fig. 1. Incoming and outgoing waves at the νth terminal pair.

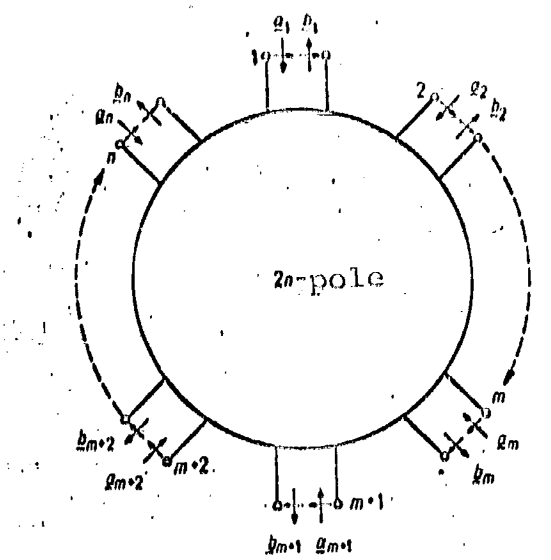

Bild 2. 2n-Pol.

Fig. 2. 2n-pole

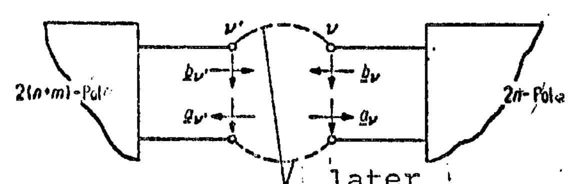

Fig. 3. Counting arrows on terminal pairs ν'

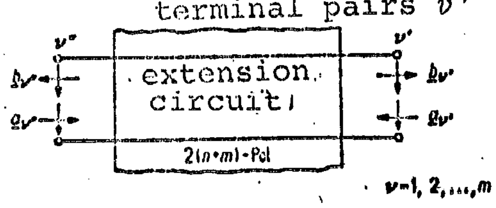

Fig. 4. Counting arrows on terminal pairs ν''

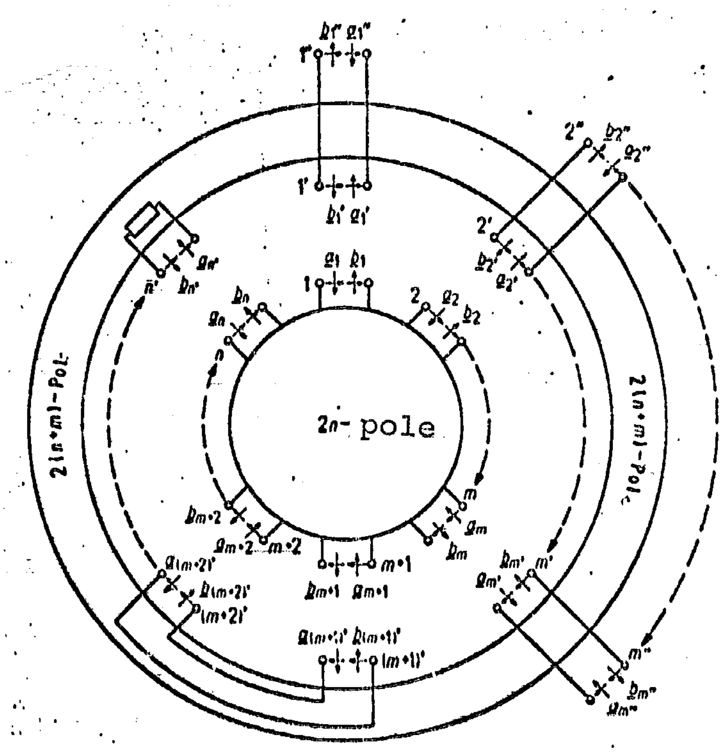

Bild 5. 2(n + m)-Pol.

Fig. 5. 2(n + m)-pole

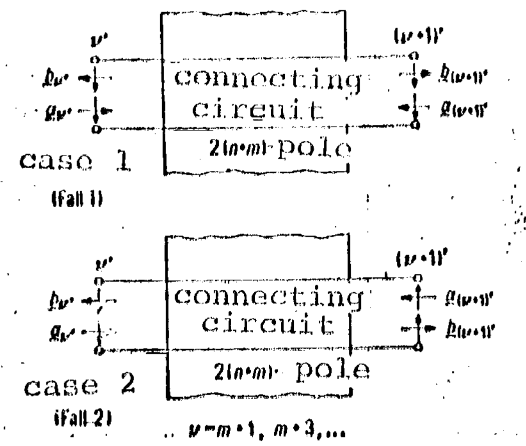

Fig. 6. Connecting circuits

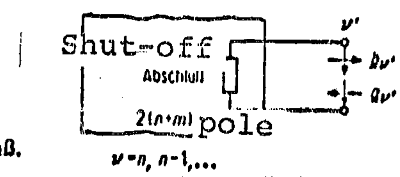

Bild 7. Abschluß.

Fig. 7. Shut-off

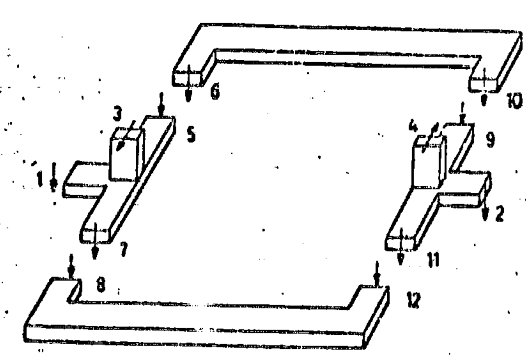

Fig. 8 Scattering matrix
for microwave bridge